Non parametric tests for Poisson processes: studies on spatial representativeness of services

JANVIER 2012

J. BESSAC(*), F. COQUET(*)(**), J.-M. FLOCH(***), M. FROMONT(*)(**)

(*) IRMAR

(**) CREST-Ensai

(***) INSEE-DDAR

Introduction

We are interested in this paper in studying the spatial representativeness of services like schools, medical services, pharmacies, shops, restaurants, banks... in the city of Rennes. More precisely, we focus on the two following questions. May two different services be assumed to be identically spatially distributed in the city or in a restricted area? Is the spatial distribution of one particular service homogeneous with respect to houses in the city or in a restricted area of interest?

Assuming that the spatial representation of houses or services can be modelized by a spatial Poisson process (see [15] and [3] for instance), these questions can be translated from a statistical point of view as problems of testing proportionality or equality of the intensities of two independent spatial Poisson processes.

Here we choose to investigate non parametric tests that have been recently proposed by Baringhaus and Franz [2], Gretton et al. [14], and Fromont, Laurent, Reynaud-Bouret [12]. Since the tests by Fromont, Laurent, Reynaud-Bouret were not studied in practice in the original paper in a spatial context, we first evaluate the performance of the above tests with multivariate simulated data.

Then, we apply them to economic data from l'INSEE containing the (x, y)-coordinates of houses and services on a map of Rennes in 2007. The obtained results are mostly in accordance with our expectations. But some of these results also pose new theoretical questions, thus confirming that modelizing economic data with Poisson processes, more frequently used in reliability and biology, offers a lot of possibilities.

1 Point processes and Poisson processes

1.1 Definition and first properties

Point processes are the mathematical tool at hand to modelize dots (or events) that appear at random on a domain. Here, "at random" may be understood with respect to time, or with respect to space, or both. Let us take the example of the opening of new shops in a given town along a specific year: we may be interested in the specific times of opening, if we want to identify periods of more or less active trading; alternatively we may consider the locations chosen by these shops if we want to get a picture of the more or less dynamic areas, or we can get interested in when and where the shops open. One mathematical way to address those problems is to represent each new shop by a dot on the map, and to observe where or/and when the dots appear: the outcome is a point process.

Poisson processes are by far the most popular point processes. Roughly speaking, a point process is a Poisson process if dots appear independently (in time and/or in space) from each other. Remarkably enough, this heuristics has led to a variety of definitions of a Poisson process, according to the field of interest, all of which are luckily compatible.

The definition we shall use here is taken from [17]. It seems to be the closest to our purpose.

A spatial point process N is a random countable subset of $\mathbb{X} \subset \mathbf{R}^2$. We associate to N its intensity, which is a measure ξ on $\mathbb{X}$, assumed to be bounded on compact sets and absolutely continuous w.r.t. the Lebesgue measure ν. Let s be the associated density, also called intensity of N. For all $B \subset X$, we denote by $N(B)$ the number of elements of N that lie in B.

Definition 1. *N is a Poisson process on $\mathbb{X} \subset \mathbb{R}^2$ with intensity s w.r.t. ν if and only if*

1. *For every $B \subset \mathbb{X}$ such that $\xi(B) < \infty$, $N(B)$ is distributed according to a Poisson distribution with parameter $\xi(B) = \int_B s(x)d\nu_x$;*
2. *Conditionally to the event "$N(B) = n$", $N \cap B$ has the same distribution as an n i.i.d. sample with common density $s/\int_B s(x)d\nu_x$ with respect to the Lebesgue measure ν on $\mathbb{X}$.*

The intensity s has to be understood in an easy way: in areas of $\mathbb{X}$ where s takes high values, you will expect more dots than in areas where s takes low values, and you will find no dots at all in subsets of $\mathbb{X}$ where $s = 0$. If the intensity s is constant, you will expect the dots to be uniformly distributed on $\mathbb{X}$.

Definition 2. *N is said to be homogeneous if and only if its intensity is constant on $\mathbb{X}$.*

Among the classical properties of a Poisson process, we only recall here the most important one.

Proposition 1. *If $B_1, \cdots, B_k$ are disjoint subsets of $\mathbb{X}$, then $N \cap B_1, \cdots, N \cap B_k$ are independent.*

1.2 Poisson processes and services or houses representativeness

Recall that we study the representativeness of services in the city of Rennes. It seems rather natural to assume that the coordinates of specific services or houses form non-homogeneous Poisson processes in a subset of $\mathbb{R}^2$.

In order to be perfectly rigorous, this assumption should of course be validated by a statistical test, but to our knowledge, there is no such test that could be used in practice yet.

Starting however from this assumption, we can compare the distributions of two services in any area of Rennes, or compare the distribution of a particular service with the distribution of houses by the means of statistical hypotheses tests of proportionality or equality of the intensities of two Poisson processes.

2 Two-sample problems for spatial Poisson processes

Let us consider a measurable subspace $\mathbb{X}$ of $\mathbb{R}^2$, equipped with the Lebesgue measure ν. Let N_1 and N_2 be two independent Poisson processes observed on $\mathbb{X}$, whose intensities with respect to ν are denoted by s_1 and s_2, and whose numbers of points are respectively denoted by $|N_1|$ and $|N_2|$. Let now $(X_1, \dots, X_{|N_1|})$ and $(Y_1, \dots, Y_{|N_2|})$ denote the points of the processes N_1 and N_2 respectively.

Given the observation of N_1 and N_2, we first address the question of testing the null hypothesis (H_0^p): "s_1 and s_2 are proportional" against the alternative (H_1^p): "they are not". Some papers deal with the problem of testing (H_0^p): "s_1/s_2 is constant" against "it is increasing", such as [5] and [10]. Though the alternative "s_1/s_2 is increasing" is usual in reliability contexts, it has no sense in our context of services representativeness study. We will use other tests, that will be called here "conditional" tests. Notice that (H_0^p) is true if and only if $s_1/\int_{\mathbb{X}} s_1(x)d\nu_x = s_2/\int_{\mathbb{X}} s_2(x)d\nu_x$. Therefore, from Definition 1 of Poisson processes, one deduces that testing (H_0^p) against (H_1^p) amounts to testing distributions equality for the two i.i.d. samples $(X_1, \dots, \dots, X_{n_1})$ and $(Y_1, \dots, Y_{n_2})$ with respective sizes n_1 and n_2 obtained when considering N_1 and N_2 conditionally to the event "$|N_1| = n_1$ and $|N_2| = n_2$". Many procedures have been and are still developed to solve this classical i.i.d. two-sample problem. Of course, we first think about the famous Kolmogorov-Smirnov and Cramer von Mises tests. However, while these tests are very simple to understand and implement when the observations are univariate, their extensions to multivariate observations are not so clear. One can actually generalize the Kolmogorov-Smirnov statistic for instance in many different ways, generally expressed in such a form:

$$T_{KS} = \sup_{\theta \in \Theta} \left| \frac{1}{n_1} \sum_{i=1}^{n_1} \theta(X_i) - \frac{1}{n_2} \sum_{j=1}^{n_2} \theta(Y_j) \right|,$$

where Θ is a particular class of measurable functions: $\mathbb{X} \to \mathbb{R}$. Note that when $\mathbb{X} = \mathbb{R}$ and Θ is the set of indicators of cells $(-\infty, t]$, this exactly reduces to the well-known Kolmogorov-Smirnov statistic. In this case, since the underlying distributions of the X_i's and the Y_j's are assumed to be atomless, this statistic is distribution free under the null hypothesis, and the critical values of the corresponding test are easy to compute. In the case where $\mathbb{X} \subset \mathbb{R}^2$, this is not so simple. First, the choice of the class Θ is not obvious. Then, the fact that the resulting statistic is in general not distribution free under the null hypothesis also poses a crucial question: which critical values can we take here?

This question is usually solved through general bootstrap approaches including Efron's bootstrap or permutation bootstrap approaches (see [19] for instance). Friedman and Rafsky proposed asymptotically distribution free multivariate extensions of Kolmogorov-Smirnov and Wald-Wolfowitz testing statistics under the null hypothesis. We chose to investigate a new version of the old Cramer test proposed by Baringhaus and Franz [2], which has appeared to be competitive in the univariate case, regarding Kolmogorov-Smirnov and Cramer von Mises tests, and the recent Kernel Maximum Mean Discrepancy test proposed by Gretton et al. [14], which has been compared to Friedman and Rafsky's test among others.

We secondly address the question of testing (H_0): "$s_1 = s_2$" against the alternative (H_1): "$s_1 \neq s_2$". Many papers deal with this two-sample problem for homogeneous Poisson processes such as, among others, the historical ones of [20], [8], [13], and [21], or the more recent ones of [16], [18], [7], and [6]. However, very few papers focus on this two-sample problem for non-homogeneous Poisson processes, which is considered here. To our knowledge, the paper by Fromont, Laurent, Reynaud-Bouret [12] is the only one to address this problem exactly. Of course, any level α test of (H_0^p) against (H_1^p) is also a level α test of (H_0) against (H_1), but the resulting test may be too conservative. Hence, when the problem of testing (H_0) against (H_1) is the only one to be considered, we exclusively investigate the tests proposed in [12].

We detail all the investigated tests in the two following sections.

Let $\mathbb{P}_{s_1,s_2}$ be the joint distribution of (N_1, N_2). We set for any event $\mathcal{A}$ based on (N_1, N_2), $\mathbb{P}_{(H_0)}(\mathcal{A}) = \sup_{s_1,s_2,s_1=s_2} \mathbb{P}_{s_1,s_2}(\mathcal{A})$.

2.1 Conditional tests from the classical i.i.d. two-sample problem

Let n_1 and n_2 be some positive integers. From Definition 1, we know that conditionally to the event "$|N_1| = n_1$ and $|N_2| = n_2$", $(X_1, \ldots, X_{|N_1|})$ and $(Y_1, \ldots, Y_{|N_2|})$ have the same distribution as two i.i.d. samples $(X_1 \ldots, X_{n_1})$ and $(Y_1, \ldots, Y_{n_2})$ with respective densities $\tilde{s}_1 = s_1 / \int_{\mathbb{X}} s_1(x) d\nu_x$ and $\tilde{s}_2 = s_2 / \int_{\mathbb{X}} s_2(x) d\nu_x$ w.r.t. the Lebesgue measure ν on $\mathbb{X}$. Given the observation of N_1 and N_2, we here consider the problem of testing (H_0^p): "s_1 and s_2 are proportional" against (H_1^p): "they are not", which amounts to testing the null hypothesis of equality between the distributions of the i.i.d. samples $(X_1 \ldots, X_{n_1})$ and $(Y_1, \ldots, Y_{n_2})$.

2.1.1 Cramer test

Baringhaus and Franz [2] start from a result stating that if $\|.\|$ denotes the Euclidean norm of $\mathbb{R}^d$, $\tilde{X}$, $\tilde{X}'$ are random vectors of $\mathbb{R}^d$ with the same density $\tilde{s}_1$ w.r.t. the Lebesgue measure, with finite expectation, $\tilde{Y}$, $\tilde{Y}'$ are random vectors of $\mathbb{R}^d$ with the same density $\tilde{s}_2$, with finite expectation, and if $\tilde{X}$, $\tilde{X}'$, $\tilde{Y}$, $\tilde{Y}'$ are independent, then

$$2\mathbb{E}\left[\|\tilde{X} - \tilde{Y}\|\right] - \mathbb{E}\left[\|\tilde{X} - \tilde{X}'\|\right] - \mathbb{E}\left[\|\tilde{Y} - \tilde{Y}'\|\right] \geq 0.$$

Moreover, the equality is true if and only if $\tilde{s}_1 = \tilde{s}_2$ (see Theorem 2.1 in [2]). This result and the law of large numbers lead to the following testing statistic:

$$\begin{aligned} T_{Cramer} &= \frac{n_1 n_2}{n_1+n_2}\left(\frac{1}{n_1 n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\|X_i - Y_j\| - \frac{1}{2n_1^2}\sum_{i,k=1}^{n_1}\|X_i - X_k\|\right. \\ &\quad \left. -\frac{1}{2n_2^2}\sum_{j,k=1}^{n_2}\|Y_j - Y_k\|\right) \\ &= \frac{n_1 n_2}{n_1+n_2}\left(\frac{2}{n_1 n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\varphi\left(\|X_i - Y_j\|^2\right) - \frac{1}{n_1^2}\sum_{i,k=1}^{n_1}\varphi\left(\|X_i - X_k\|^2\right)\right. \\ &\quad \left. -\frac{1}{n_2^2}\sum_{j,k=1}^{n_2}\varphi\left(\|Y_j - Y_k\|^2\right)\right), \end{aligned}$$

with $\varphi(t) = \sqrt{t}/2$. The authors then suggest to reject the null hypothesis (H_0^p) when T_{Cramer} is large, that is in fact larger than a critical value to define. Contrary to the classical Kolmogorov-Smirnov or Cramer-von Mises testing statistics in the univariate context, and to Friedman-Rafsky testing statistic in the multivariate context, the statistic T_{Cramer} is not distribution free under the null hypothesis. Hence, given a prescribed level α, the $(1-\alpha)$ quantile of T_{Cramer} under (H_0^p) can not be taken as critical value for the test. Baringhaus and Franz propose to consider either an Efron's bootstrapped or a permutation bootstrapped version T^*_{Cramer} of the statistic T_{Cramer} given the pooled sample $Z = (X_1, \ldots, X_{n_1}, Y_1, \ldots, Y_{n_2})$. As explained in details in [23] for instance, they consider the $(1-\alpha)$ quantile of T^*_{Cramer} given Z, that we denote here by $c^*_{Cramer}(1-\alpha)$. The test proposed by Baringhaus and Franz [2] then consists in rejecting the null hypothesis (H_0^p) when T_{Cramer} is larger than $c^*_{Cramer}(1-\alpha)$. Let us introduce the corresponding test function, that we denote by Φ_{Cramer}:

$$\Phi_{Cramer} = \mathbf{1}_{T_{Cramer} > c^*_{Cramer}(1-\alpha)}. \tag{2.1}$$

Baringhaus and Franz call their test Cramer test as Cramer [9] already proposed a similar testing statistic for the one-sample goodness-of-fit problem in the univariate context. They use asymptotic arguments from [23] to validate the bootstrap approach. They thus prove that their test is asymptotically of level α, and that it is consistent against any fixed alternative. They finally estimate the powers of their test mainly under univariate and multivariate normal location and dispersion alternatives, and they compare these powers with the ones of the usual parametric t-test and F-test, Kolmogorov-Smirnov and Cramer von Mises tests in the univariate context, and with the ones of the Hotelling's T^2-test and Bartlett's LR-test in the multivariate contexts. They conclude that Cramer test performs well under such alternatives.

The test Φ_{Cramer} is furthermore implemented in the software environment `R` in the function `cramer.test` of the `cramer` package, where various options are available. Among these options, the user can choose to apply the test Φ_{Cramer} defined in (2.1), but also to apply another version of the test which is defined in the same way as in (2.1), just changing φ in the second expression of T_{Cramer}. We will only consider in the following two cases: the original test denoted by Φ_{Cramer}, and the test corresponding to the choice of $\varphi(t) = 1 - \exp(-t/2)$ which was proposed by Bahr [1] and which is hence defined by

$$\Phi_{Bahr} = \mathbf{1}_{T_{Bahr} > c^*_{Bahr}(1-\alpha)}, \tag{2.2}$$

where

$$T_{Bahr} = \frac{n_1 n_2}{n_1+n_2}\left(\frac{2}{n_1 n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}\varphi\left(\|X_i-Y_j\|^2\right)-\frac{1}{n_1^2}\sum_{i,k=1}^{n_1}\varphi\left(\|X_i-X_k\|^2\right)\right.$$
$$\left.-\frac{1}{n_2^2}\sum_{j,k=1}^{n_2}\varphi\left(\|Y_j-Y_k\|^2\right)\right),$$

with $\varphi(t) = 1-\exp(-t/2)$, and $c^*_{Bahr}(1-\alpha)$ is a critical value obtained from a bootstrap approach. The user may also choose either Efron's bootstrap or permutation bootstrap to determine the critical values of the tests.

2.1.2 Kernel Maximum Mean Discrepancy test

Let Θ be a class of functions $\theta : \mathbb{X} \to \mathbb{R}$ and let $\tilde{X}$ and $\tilde{Y}$ be two independent random variables on $\mathbb{X}$ with respective densities $\tilde{s}_1$ and $\tilde{s}_2$ w.r.t. ν. Gretton et al. [14] define the Maximum Mean Discrepancy (MMD) over Θ as:

$$MMD[\Theta,\tilde{s}_1,\tilde{s}_2] = \sup_{\theta\in\Theta}\left(\mathbb{E}_{\tilde{s}_1}[\theta(\tilde{X})]-\mathbb{E}_{\tilde{s}_2}[\theta(\tilde{Y})]\right) \tag{2.3}$$

$$= \sup_{\theta\in\Theta}\left(\int_{\mathbb{X}}\theta(x)\tilde{s}_1(x)d\nu_x-\int_{\mathbb{X}}\theta(y)\tilde{s}_2(y)d\nu_y\right). \tag{2.4}$$

Noticing that when Θ is the space of bounded continuous functions on $\mathbb{X}$, $MMD[\Theta,\tilde{s}_1,\tilde{s}_2] = 0$ if and only if $\tilde{s}_1 = \tilde{s}_2$, any "good" estimator of $MMD[\Theta,\tilde{s}_1,\tilde{s}_2]$ or $MMD[\Theta,\tilde{s}_1,\tilde{s}_2]^2$ for instance may be a pertinent testing statistic. Since one can reasonably not work in practice with the space of bounded continuous functions on $\mathbb{X}$, Gretton et al. [14] suggest to consider other classes of functions Θ, which are rich enough to guarantee that $MMD[\Theta,\tilde{s}_1,\tilde{s}_2] = 0$ if and only if $\tilde{s}_1 = \tilde{s}_2$, but restrictive enough for the resulting test to be consistent. Namely, they consider the unit balls of universal Reproducing Kernel Hilbert Spaces. Universal RKHSs are defined in [14] and it is proved in particular in [22] that the RKHS associated with the usual Gaussian kernel is universal. The authors actually prove that when Θ is the unit ball of such a universal RKHS $\mathcal{H}_K$ defined on $\mathbb{X}$ with associated positive definite kernel K and representation function ψ, then the equivalence $MMD[\Theta,\tilde{s}_1,\tilde{s}_2] = 0 \Leftrightarrow \tilde{s}_1 = \tilde{s}_2$ holds. Moreover, from a lemma in [4], one has in this case that

$$MMD[\Theta,\tilde{s}_1,\tilde{s}_2]^2 = \|\mathbb{E}_{\tilde{s}_1}[\psi(\tilde{X})]-\mathbb{E}_{\tilde{s}_2}[\psi(\tilde{Y})]\|^2_{\mathcal{H}_K}$$
$$= \mathbb{E}_{\tilde{s}_2}K(\tilde{X},\tilde{X}')+\mathbb{E}_{\tilde{s}_2}K(\tilde{Y},\tilde{Y}')-2\mathbb{E}_{\tilde{s}_1,\tilde{s}_2}K(\tilde{X},\tilde{Y}),$$

where $\tilde{X}'$ and $\tilde{Y}'$ are independent copies of $\tilde{X}$ and $\tilde{Y}$, independent from $\tilde{X}$ and $\tilde{Y}$, and $\|.\|_{\mathcal{H}_K}$ is the norm in $\mathcal{H}_K$. Hence an unbiased estimator of $MMD[\Theta,\tilde{s}_1,\tilde{s}_2]^2$ is easily obtained when $n_1 = n_2$ as

$$T_{KMMD,n_1} = \frac{1}{n_1(n_1-1)}\sum_{i\neq j=1}^{n_1}\Big(K(X_i,X_j)+K(Y_i,Y_j)-K(X_i,Y_j)-K(X_j,Y_i)\Big). \tag{2.5}$$

One can also always (that is also when $n_1 \neq n_2$) consider the empirical estimator of $MMD[\Theta,\tilde{s}_1,\tilde{s}_2]$ defined by

$$T_{KMMD,n_1,n_2} = \left(\frac{1}{n_1^2}\sum_{i,k=1}^{n_1}K(X_i,X_k)+\frac{1}{n_2^2}\sum_{j,k=1}^{n_2}K(Y_j,Y_k)-\frac{2}{n_1 n_2}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}K(X_i,Y_j)\right)^{\frac{1}{2}}, \tag{2.6}$$

and choose one of these estimators as testing statistic. Given a prescribed level α, Gretton et al. [14] propose to reject the null hypothesis (H_0^p) when T_{KMMD,n_1} is larger than a critical value $c^*_{KMMD,n_1}(1-\alpha)$ or when T_{KMMD,n_1,n_2} is larger than $c^*_{KMMD,n_1,n_2}(1-\alpha)$. $c^*_{KMMD,n_1}(1-\alpha)$ and $c^*_{KMMD,n_1,n_2}(1-\alpha)$ may be deduced from different approaches. $c^*_{KMMD,n_1}(1-\alpha)$ may be determined from a uniform convergence bound for T_{KMMD,n_1} based on Hoeffding's concentration inequality under the null hypothesis or from an estimation of the $(1-\alpha)$ asymptotic quantile of T_{KMMD,n_1} under the null hypothesis based on either Efron's bootstrap approach given the pooled sample Z or a moments approximation approach. $c^*_{KMMD,n_1,n_2}(1-\alpha)$ may also be determined from a uniform convergence bound for T_{KMMD,n_1,n_2} based on Hoeffding's concentration inequality under the null hypothesis, or a bootstrap approach.

Let us introduce the corresponding test functions:

$$\Phi_{KMMD,n_1} = \mathbf{1}_{T_{KMMD,n_1} > c^*_{KMMD,n_1}(1-\alpha)}, \tag{2.7}$$

and

$$\Phi_{KMMD,n_1,n_2} = \mathbf{1}_{T_{KMMD,n_1,n_2} > c^*_{KMMD,n_1,n_2}(1-\alpha)}. \tag{2.8}$$

Note that when K is the usual Gaussian kernel with a bandwidth equal to 1 that is when $K(x,x') = \exp(-\|x-x'\|^2/2)$, and when the critical value $c^*_{KMMD,n_1,n_2}(1-\alpha)$ is obtained from a bootstrap method, Φ_{KMMD,n_1,n_2} is very close to Φ_{Bahr}. Indeed, in this case,

$$T^2_{KMMD,n_1,n_2} = \frac{n_1+n_2}{n_1 n_2} T_{Bahr}.$$

2.2 Adaptive non parametric multiple testing procedures

Let us now focus ont he problem of testing (H_0): "$s_1 = s_2$" against the alternative (H_1): "$s_1 \neq s_2$". We here give a short description of the testing procedures proposed by Fromont, Laurent, Reynaud-Bouret[12]. For more details, we refer to the original paper.

We denote by $\|.\|_\nu$ the $\mathbb{L}^2(\mathbb{X}, d\nu)$-norm, and by $< .,. >_\nu$ the scalar product associated with $\|.\|_\nu$ on $\mathbb{X}$.

We assume as in [12] that s_1 and s_2 are both in $\mathbb{L}^\infty(\mathbb{X}) \cap \mathbb{L}^1(\mathbb{X}, d\nu)$. Noticing that this in particular implies that s_1 and s_2 belong to $\mathbb{L}^2(\mathbb{X}, d\nu)$, Fromont, Laurent, Reynaud-Bouret propose to use non parametric estimators of $\|f-g\|^2_\nu$ as testing statistics. For instance, considering a finite dimensional subspace S of $\mathbb{L}^2(\mathbb{X}, d\nu)$ and an orthonormal basis $\{\varphi_\lambda, \lambda \in \Lambda\}$ of S for $< .,. >_\nu$, they introduce the random variable $\hat{T}$ defined by

$$\hat{T} = \sum_{\lambda \in \Lambda} \sum_{i \neq i'=1}^{|N|} \varphi_\lambda(Z_i)\varphi_\lambda(Z_{i'})\varepsilon_i^0 \varepsilon_{i'}^0,$$

where $(Z_1, \ldots, Z_{|N|})$ denotes the points of the pooled Poisson process N composed of the points from both N_1 and N_2 (with size $|N| = |N_1| + |N_2|$), $\varepsilon_i^0 = 1$ when Z_i belongs to N_1 and $\varepsilon_i^0 = -1$ when Z_i belongs to N_2. Then $\hat{T}$ is an unbiased estimator of $\|\Pi_S(f-g)\|^2_\nu$, where Π_S is the orthogonal projection onto S, and it may be a relevant choice of testing statistic. Introducing the corresponding wild bootstrapped statistic defined as

$$\hat{T}^\varepsilon = \sum_{\lambda \in \Lambda} \sum_{i \neq i'=1}^{|N|} \varphi_\lambda(Z_i)\varphi_\lambda(Z_{i'})\varepsilon_i \varepsilon_{i'}$$

where $(\varepsilon_i)_{i\in\mathbb{N}}$ is a sequence of i.i.d. Rademacher variables independent of N, Fromont, Laurent, Reynaud-Bouret prove that conditionally to N, under (H_0), the distribution of $\hat{T}^\varepsilon$ is exactly the same as the distribution of $\hat{T}$. Hence, given a prescribed level α in $(0,1)$, they introduce the $(1-\alpha)$ quantile of $\hat{T}^\varepsilon$ conditionally to N denoted by $q_{1-\alpha}^{(N)}$ and they reject (H_0) when $\hat{T} > q_{1-\alpha}^{(N)}$. The conditional quantile $q_{1-\alpha}^{(N)}$ hence presents the following advantages: first it leads to an exact level α test, and secondly, it can be easily approximated by a classical Monte Carlo method.

Starting from this first procedure, Fromont, Laurent, Reynaud-Bouret generalize it by noticing that the function: $\mathbb{X}^2 \to \mathbb{R}$, $(z,z') \mapsto \sum_{\lambda\in\Lambda} \varphi_\lambda(z)\varphi_\lambda(z')$ is known as a projection kernel in Learning Theory. They propose then to consider any symmetric kernel function K chosen among the three possibilities described below, and to introduce the testing statistic

$$\hat{T}_K = \sum_{i\neq i'=1}^{|N|} K(Z_i, Z_{i'})\varepsilon_i^0\varepsilon_{i'}^0. \tag{2.9}$$

As above, considering its wild bootstrapped version

$$\hat{T}_K^\varepsilon = \sum_{i\neq i'=1}^{|N|} K(Z_i, Z_{i'})\varepsilon_i\varepsilon_{i'}, \tag{2.10}$$

and the $(1-\alpha)$ quantile of $\hat{T}_K^\varepsilon$ conditionally to N that they denote by $q_{K,1-\alpha}^{(N)}$, they propose to reject (H_0) when

$$\hat{T}_K > q_{K,1-\alpha}^{(N)}.$$

Let us give below the three possible choices for K.

1. A first choice for K is a symmetric kernel function based on an orthonormal family:

$$K(z,z') = \sum_{\lambda\in\Lambda} \varphi_\lambda(z)\varphi_\lambda(z'),$$

   where $\{\varphi_\lambda, \lambda \in \Lambda\}$ is an orthonormal family for $< .,. >_\nu$.

2. A second choice for K is a kernel function based on an approximation kernel k: for $z = (z_1, z_2)$, $z' = (z'_1, z'_2)$ in $\mathbb{R}^2$,

$$K(z,z') = \frac{1}{h_1 h_2} k\left(\frac{z_1 - z'_1}{h_1}, \frac{z_2 - z'_2}{h_2}\right),$$

   where k is an approximation kernel in $\mathbb{L}^2(\mathbb{R}^2)$, and such that $k(-z) = k(z)$, and $h = (h_1, h_2)$ is a vector of 2 positive bandwiths.

3. A third choice for K is a learning or Mercer kernel such that

$$K(z,z') = \langle \psi(z), \psi(z') \rangle_{\mathcal{H}_K},$$

   where ψ and $\mathcal{H}_K$ are respectively a representation function and a RKHS associated with K. Here $\langle .,. \rangle_{\mathcal{H}_K}$ denotes the scalar product of $\mathcal{H}_K$.

This test could be viewed as a version of the test of Gretton et al. [14] adapted to the Poisson framework. However, the critical value is here not chosen in the same way as in [14]. Fromont, Laurent, Reynaud-Bouret [12] actually propose a critical value leading to a test with good theoretical non asymptotic performance in the sense that it is exactly of level α and that it also achieves a prescribed probability of second kind error for s_1 and s_2 that are not very far from each other.

Of course, the question of the choice of the kernel function K is crucial here, since different kernel functions may lead to very different performance. While Gretton et al. [14] calibrate the parameters of the chosen kernel function by a heuristic approach, Fromont, Laurent, Reynaud-Bouret [12] overcome this difficulty by considering an aggregation method specific to adaptive testing. Instead of taking a single kernel function, they propose to introduce a finite collection of kernel functions $\{K_m, m \in \mathcal{M}\}$, chosen among the possibilities listed above. For every m in $\mathcal{M}$, let $\hat{T}_{K_m}$ and $\hat{T}^{\varepsilon}_{K_m}$ be respectively defined by (2.9) and (2.10) with $K = K_m$, and let $\{w_m, m \in \mathcal{M}\}$ be a collection of positive numbers such that $\sum_{m\in\mathcal{M}} e^{-w_m} \leq 1$. For $u \in (0,1)$, let $q^{(N)}_{m,1-u}$ be the $(1-u)$ quantile of $\hat{T}^{\varepsilon}_{K_m}$ conditionally to the pooled process N. The test proposed in [12] rejects (H_0) when there exists at least one m in $\mathcal{M}$ such that

$$\hat{T}_{K_m} > q^{(N)}_{m,1-u^{(N)}_\alpha e^{-w_m}}$$

where $u^{(N)}_\alpha$ is defined as

$$u^{(N)}_\alpha = \sup\left\{u > 0, \mathbb{P}\left(\sup_{m\in\mathcal{M}} (\hat{T}^{\varepsilon}_{K_m} - q^{(N)}_{m,1-ue^{-w_m}}) > 0 \,\middle|\, N\right) \leq \alpha\right\}. \tag{2.11}$$

Let Φ_{Agg} be the corresponding test function defined by

$$\Phi_{Agg} = \mathbf{1}_{\sup_{m\in\mathcal{M}}\left(\hat{T}_{K_m} - q^{(N)}_{m,1-u^{(N)}_\alpha e^{-w_m}}\right) > 0}. \tag{2.12}$$

Note that given the observation of the pooled process N, $u^{(N)}_\alpha$ and the quantiles $q^{(N)}_{m,1-u^{(N)}_\alpha e^{-w_m}}$ can be estimated by a Monte Carlo procedure.

This multiple testing procedure has been constructed to be exactly of level α that is

$$\mathbb{P}_{(H_0)}\left(\Phi_{Agg} = 1\right) \leq \alpha.$$

Then it is proved to satisfy oracle type inequalities, and when adequate approximation kernels are chosen, to be adaptive in the minimax sense over multivariate Sobolev and anisotropic Nikol'skii-Besov balls.

2.3 Simulation study

We aim here at evaluating the practical performance in terms of levels and powers of the four tests Φ_{Cramer}, Φ_{Bahr}, Φ_{KMMD,n_1,n_2} and Φ_{Agg} defined by (2.1), (2.2), (2.8) and (2.12). We consider various densities with respect to the Lebesgue measure on $\mathbb{X} = [0,1]^2$ or $\mathbb{X} = \mathbb{R}^2$, that are uniform and normal densities, and deviations from them. Let us thus introduce the following notations:

$$\begin{aligned} f_{a,\varepsilon}(x) &= \mathbf{1}_{(0,1)^2}(x) + \varepsilon \mathbf{1}_{(0,a)^2}(x) - \varepsilon \mathbf{1}_{(a,2a)^2}(x), \\ f_\mu(x) &= \frac{1}{2\times 0.15^2\pi} \exp\left(\|x-\mu\|^2/(2\times 0.15^2)\right). \end{aligned}$$

We first choose several parameters (a, ε) in $[0, 0.5] \times [0, 1]$ and μ in $\mathbb{R}$ and we realize 1000 simulations of two independent Poisson Processes N_1 and N_2 with respective intensities $200 f_{0,0}$ and $200 f_{a,\varepsilon}$ or $200 f_{0.5}$ and $200 f_\mu$ w.r.t. the Lebesgue measure ν of $\mathbb{R}^2$.

All the tests considered here are applied with a level $\alpha = 0.05$.

Working conditionally to the event "$|N_1| = n_1$ and $|N_2| = n_2$", we consider Φ_{Cramer} and Φ_{Bahr} with Efron's bootstrap method with 1000 bootstrap replicates, which are implemented in the R function `cramer.test` of the package `cramer`. Then, we consider Φ_{KMMD,n_1,n_2} with a Gaussian kernel with bandwidth σ selected from a heuristic approach of Gretton et al. [14], and we run it with the Matlab program given in open access by the authors.

We furthermore consider Φ_{Agg} with a family of kernel functions first based on the standard Gaussian approximation kernel k_G and secondly based on the Epanechnikov approximation kernel k_E, where: $k_G(z) = \exp(-\|z\|^2/2)$ for all $z \in \mathbb{R}^2$ and $k_E(z_1, z_2) = (1 - z_1^2)(1 - z_2^2)\mathbf{1}_{\|(z_1,z_2)\| \leq 1}$. For both tests, we consider the collection of bandwidths $\{h_m, m \in \mathcal{M}\} = \{1/24, 1/16, 1/12, 1/8, 1/4, 1/2\}$ and the associated collection of kernel functions $\{K_m, m \in \mathcal{M}\}$ given for all $m \in \mathcal{M}$ either by $K_m(z, z') = \frac{1}{h_m^2} k_G\left(\frac{z-z'}{h_m}\right)$ or $K_m(z, z') = \frac{1}{h_m^2} k_E\left(\frac{z-z'}{h_m}\right)$. We take for both tests $w_m = 1/|\mathcal{M}| = 1/6$, and we denote them respectively by $\Phi_{Agg,G}$ and $\Phi_{Agg,E}$. Let us recall that the tests $\Phi_{Agg,G}$ and $\Phi_{Agg,E}$ reject the null hypothesis (H_0) when there exists m in $\mathcal{M}$ such that

$$\hat{T}_{K_m} \geq q^{(N)}_{m,1-u_\alpha^{(N)} e^{-w_m}}$$

where N corresponds to the pooled process obtained from N_1 and N_2, and $u_\alpha^{(N)}$ is defined in (2.11). Hence, for each observation of the pooled process N, we have to estimate $u_\alpha^{(N)}$ and the quantiles $q^{(N)}_{m,1-u_\alpha^{(N)} e^{-w_m}}$. These estimations are done as in [12] by classical Monte Carlo methods based on the simulation of 400000 independent samples of size $|N|$ of i.i.d. Rademacher variables. Half of the samples is used to estimate the distribution of each $\hat{T}^\varepsilon_{K_m}$. The other half is used to approximate the conditional probabilities occurring in (2.11). The point $u_\alpha^{(N)}$ is obtained by dichotomy, such that the estimated conditional probability occurring in (2.11) is less than α, but as close as possible to α. We implemented the test with Matlab from Fromont, Laurent, Reynaud-Bouret [12]'s programs.

For each of the 1000 simulations of the Poisson Processes N_1 and N_2, we determine the conclusions of the tests Φ_{Cramer}, Φ_{Bahr}, Φ_{KMMD,n_1,n_2}, $\Phi_{Agg,G}$ and $\Phi_{Agg,E}$. The levels or powers of the tests are estimated by the number of rejections for these tests divided by 1000. The results are given in the following table.

Densities	Φ_{Cramer}	Φ_{Bahr}	Φ_{KMMD,n_1,n_2}	$\Phi_{Agg,G}$	$\Phi_{Agg,E}$
$(f_{0,0}, f_{0,0})$	0.052	0.052	0.06	0.0485	0.046
$(f_{0,0}, f_{0.25,0.8})$	0.10	0.09	0.15	0.17	0.18
$(f_{0,0}, f_{0.25,0.9})$	0.10	0.09	0.18	0.23	0.21
$(f_{0,0}, f_{0.25,1})$	0.14	0.11	0.21	0.26	0.25
$(f_{0.5}, f_{0.5})$	0.048	0.046	0.043	0.0485	0.046
$(f_{0.5}, f_{0.52})$	0.36	0.37	0.26	0.21	0.18
$(f_{0.5}, f_{0.54})$	0.90	0.91	0.83	0.69	0.66

Comments. The first thing we can notice here is that the tests Φ_{Cramer}, Φ_{Bahr}, and Φ_{KMMD,n_1,n_2} are more powerful than $\Phi_{Agg,G}$ and $\Phi_{Agg,E}$ when the alternative is composed of intensities with a smooth difference. On the contrary, $\Phi_{Agg,G}$ and $\Phi_{Agg,E}$ are more powerful than Φ_{Cramer}, Φ_{Bahr}, and Φ_{KMMD,n_1,n_2} when the alternative is composed of intensities with very localized differences. We are not really surprised here since the tests developed by Fromont, Laurent and Reynaud-Bouret were precisely constructed to be adaptive over classes of possibly very irregular alternatives. We could surely improve the powers of these tests for alternatives composed of intensities with smooth differences by considering another family of bandwidths or even another family of kernels.

3 Studies on spatial representativeness of services

The data we study here are reported in two tables. The first table consists in the (x, y)-coordinates on a map of the city of Rennes of all houses, and the number of flats per houses (when needed) in 2007. The second table consists in the (x, y)-coordinates on the same map of Rennes of various services such as administrative offices, shops, schools, artisans, restaurants, medical services, social services, cultural services, with an INSEE code ("A101",...,"D237") giving the precise type of each service.

3.1 Comparisons for various pairs of services

We first aim at comparing the distributions of various services. For sake of simplicity, we rescale the (x, y) original coordinates, so that $(x, y) \in [0, 1]^2$. We represent the points thus obtained for services on the following figure.

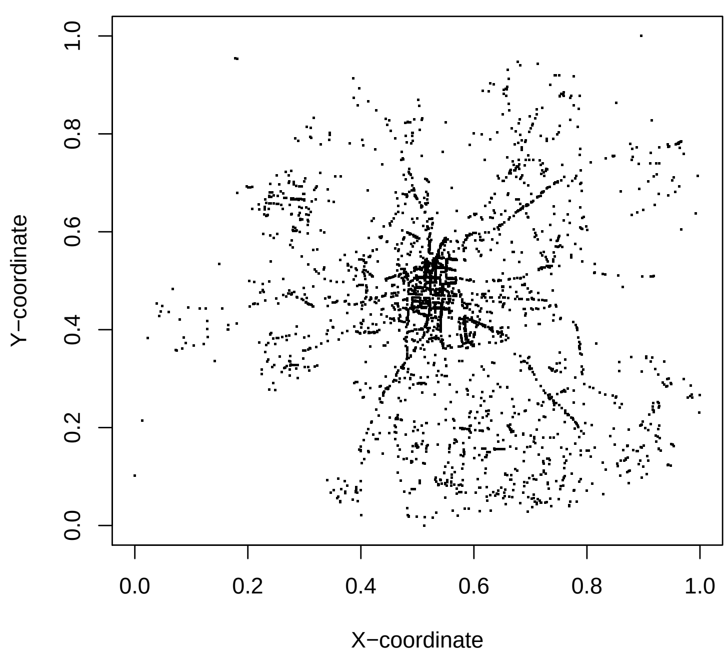

Figure 1: Representation of all services

We now consider various pairs of services, where each pair is denoted by *Service1* and *Service2*. The points defined by the rescaled coordinates corresponding to *Service1* are assumed to be a Poisson process N_1 with intensity s_1 with respect to the Lebesgue measure on $\mathbb{X} = [0, 1]^2$, and the points defined by the rescaled coordinates corresponding to *Service2* are assumed to be a Poisson process N_2 with intensity s_2 with respect to the Lebesgue measure on $\mathbb{X}$.

The points of the Poisson processes corresponding to each considered pair of services are represented in the following figures.

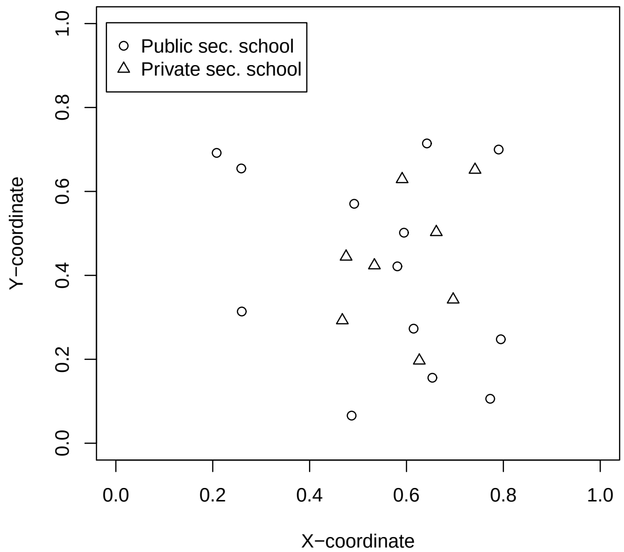

Figure 2: Representation of public and private secondary schools

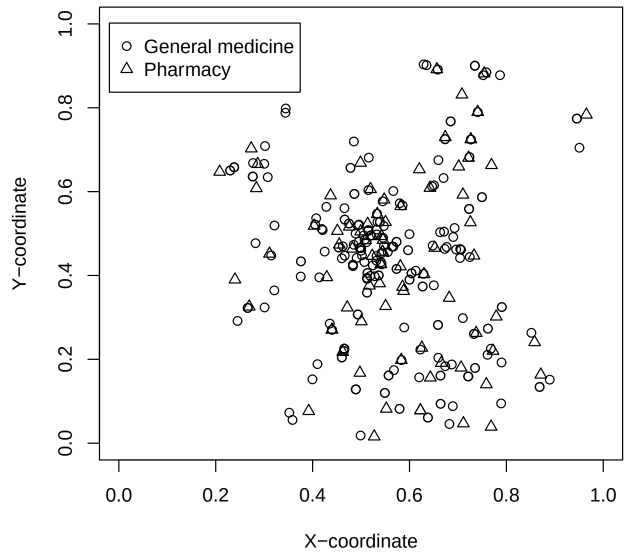

Figure 3: Representation of general medicine doctors and pharmacies

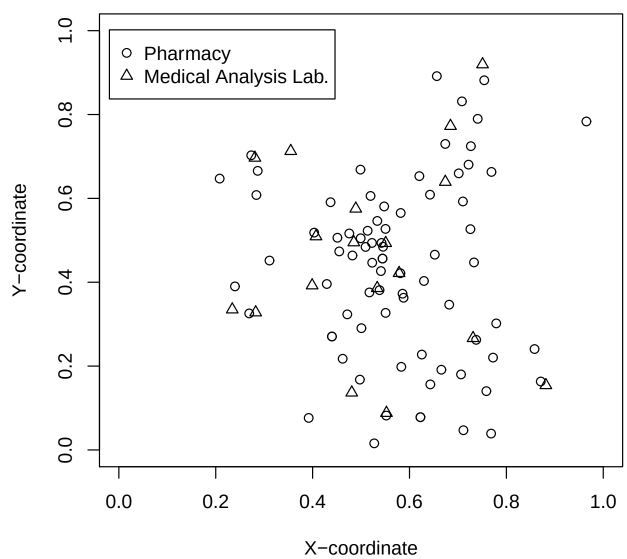

Figure 4: Representation of pharmacies and medical analysis laboratories

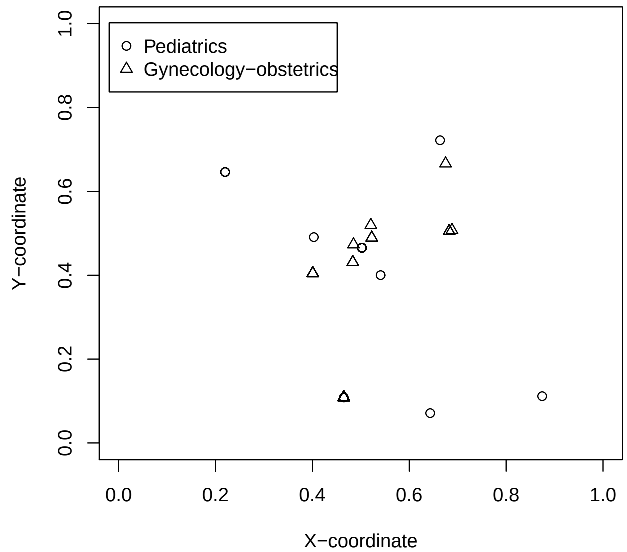

Figure 5: Representation of pediatricians and obstetricians

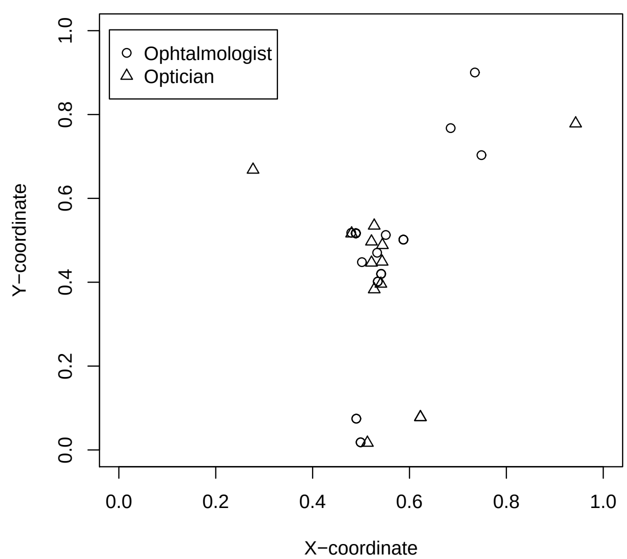

Figure 6: Representation of ophtalmologists and opticians

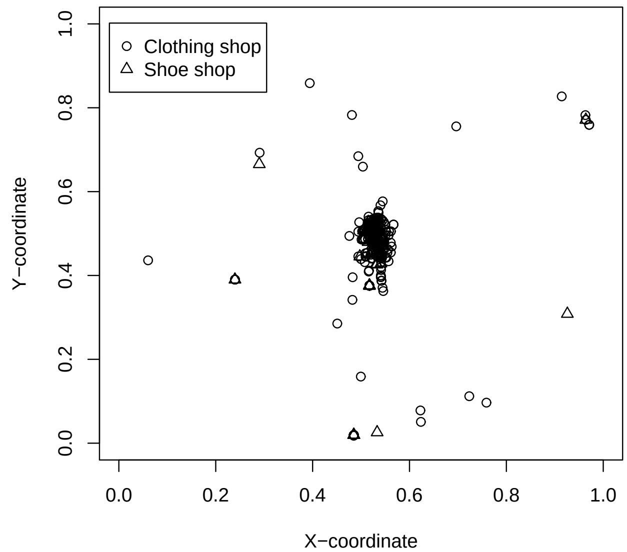

Figure 7: Representation of clothing and shoe shops

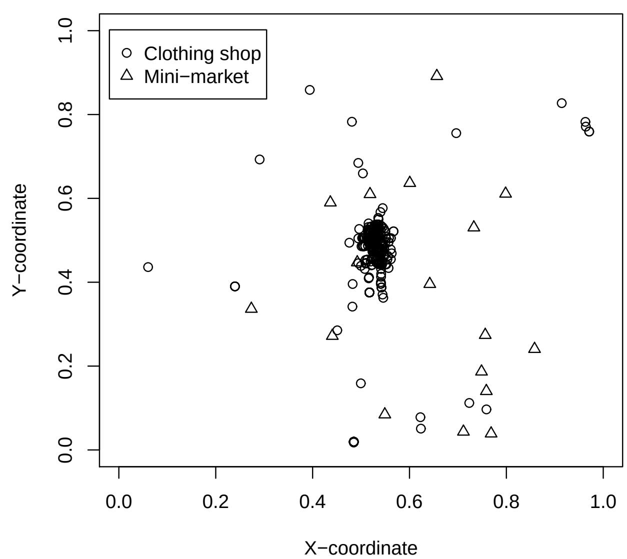

Figure 8: Representation of clothing shops and mini-markets

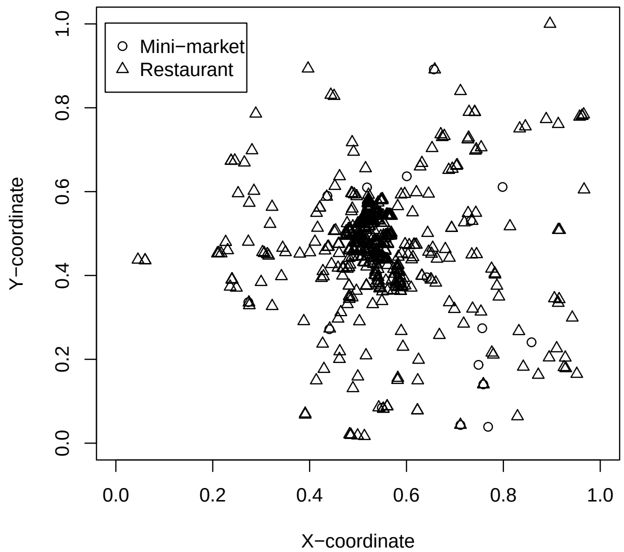

Figure 9: Representation of mini-markets and restaurants

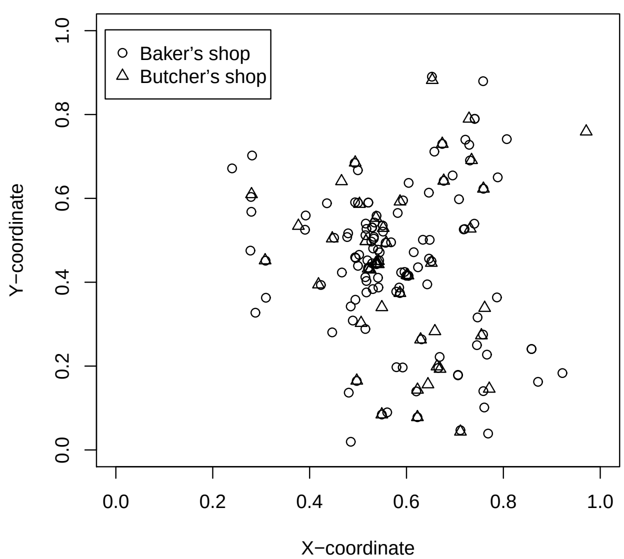

Figure 10: Representation of baker's and butcher's shops

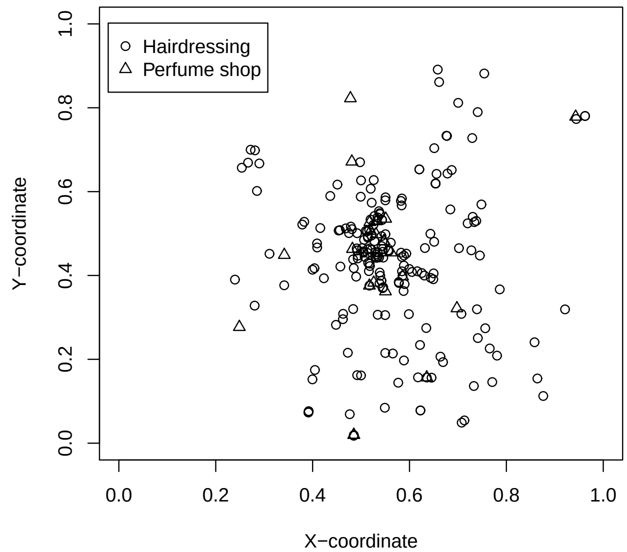

Figure 11: Representation of hairdressers and perfume shops

3.1.1 Conditional tests

In this section, we focus on the problem of testing (H_0^p) "s_1 and s_2 are proportional" against (H_1^p) "they are not" for the above pairs of services. We apply the conditional tests which are described in Section 2.1 and studied in Section 2.3.

For each considered pair of services, we work here conditionally to the event "$|N_1| = n_1$ and $|N_2| = n_2$". The tests Φ_{Cramer} and Φ_{Bahr} defined by (2.1) and (2.2) are applied with a level $\alpha = 0.05$, and with an Efron's bootstrap approximation with 100000 bootstrap replicates. The test Φ_{KMMD,n_1,n_2} defined by (2.8) is applied with a level $\alpha = 0.05$, with the method of moments approximation as recommended by the authors for moderately small data sets, and with the default Gaussian kernel (whose bandwidth σ is heuristically selected). The results are given in the following table. Each row corresponds to a pair of services. Rejection of the null hypothesis is denoted by 1, while acceptance is denoted by 0. Estimations of the p-values of the tests Φ_{Cramer} and Φ_{Bahr} are given between parentheses, and the heuristic choice of σ is given between brackets for the test Φ_{KMMD,n_1,n_2}.

Services	Φ_{Cramer}	Φ_{Bahr}	Φ_{KMMD,n_1,n_2}
Public (C201) / Private (C202) second. schools	0(0.60)	0(0.75)	$0[\sigma = 0.22]$
General medicine (D201) / Pharmacy (D301)	0(0.99)	0(0.89)	$0[\sigma = 0.21]$
Pharmacy (D301) / Medical analysis lab. (D302)	0(0.64)	0(0.46)	$0[\sigma = 0.23]$
Pediatrics (D210) / Gynecology-obstetrics (D205)	0(0.09)	0(0.21)	$0[\sigma = 0.23]$
Ophtalmologist (D208) / Optician (D234)	0(0.60)	0(0.71)	$0[\sigma = 0.18]$
Clothing shop (B302) / Shoe shop (B304)	0(0.39)	0(0.31)	$0[\sigma = 0.05]$
Clothing shop (B302) / Mini-market (B201)	1(0)	1(0)	$1[\sigma = 0.22]$
Mini-market (B201) / Restaurant (A504)	1(0)	1(0)	$1[\sigma = 0.23]$
Baker's (B203) / Butcher's (B204) shops	0(0.99)	0(0.95)	$0[\sigma = 0.18]$
Hairdressing (A501) / Perfume shop (B310)	0(0.53)	0(0.57)	$0[\sigma = 0.17]$

3.1.2 Adaptive tests

We focus now on the problem of testing (H_0) "$s_1 = s_2$" against (H_1) "$s_1 \neq s_2$" for the above pairs of services. We apply the tests $\Phi_{agg,G}$ and $\Phi_{agg,E}$ described in Section 2.2 and also studied in Section 2.3.

The results are given in the following table. Each row corresponds to a pair of services. Rejection of the null hypothesis for a level of significance $\alpha = 0.05$ is denoted by 1, while acceptance is denoted by 0.

Services	$\Phi_{agg,G}$	$\Phi_{agg,E}$
Public (C201) / Private (C202) second. schools	0	0
General medicine (D201) / Pharmacy (D301)	1	1
Pharmacy (D301) / Medical analysis lab. (D302)	1	1
Pediatrics (D210) / Gynecology-obstetrics (D205)	1	1
Ophtalmologist (D208) / Optician (D234)	0	0
Clothing shop (B302) / Shoe shop (B304)	1	1
Clothing shop (B302) / Mini-market (B201)	1	1
Mini-market (B201) / Restaurant (A504)	1	1
Baker's (B203) / Butcher's (B204) shops	1	1
Hairdressing (A501) / Perfume shop (B310)	1	1

3.1.3 Comments

We mainly obtain results that are in accordance with the intuition we could have looking at the representations of the considered services pairs. But notice that in some cases, the sizes of the Poisson processes corresponding to both services are small or moderately small, though the conditional tests considered here were essentially validated by asymptotical arguments. In other cases, the sizes of the Poisson processes are very different, with a rather small one, and the tests seem to be sensitive to such sizes considerations. These are the main limits of the present study. Surely, our considering the data as Poisson processes on the whole space $[0, 1]^2$, and comparison tests for such processes, will have to be revised. The particular structure of the data, that are distributed on a discrete network of addresses of a network of streets will have to be taken into account. The tests developed by Gretton et al. and Fromont, Laurent, Reynaud-Bouret are based on kernel functions. Kernel functions are precisely some of the most famous current tools to tackle data with particular structures. For instance, considering the same tests as above, but with a kernel specifically adapted to the structure involved here (as done for instance for graphs, trees, or images analysis) will be an interesting track to explore. A fundamental step for the construction of the test will also consist in choosing the best mathematical structure to modelize the data space.

3.2 Representativeness of various services with respect to houses

The question we tackle in this section is: "Are services well distributed with respect to the population in some particular areas of the city of Rennes?". A natural idea to answer this question is to use conditional tests as explained above, to test (H_0^p) against (H_1^p) for two Poisson processes N_1 and N_2, where N_1 modelizes the coordinates of houses while N_2 modelizes the coordinates of a service.

The main drawbacks of such a use of conditional tests are the following ones. First, we cannot here take into account the number of accommodations per address since a Poisson process can not contain several times the same point. Secondly, in much areas of the city, services are far less numerous than houses, and we have seen that it can be a drawback for the use of conditional tests. We therefore choose to consider here only a small area of the center of Rennes, where several services may be rather numerous.

Let us represent some of these services with respect to houses.

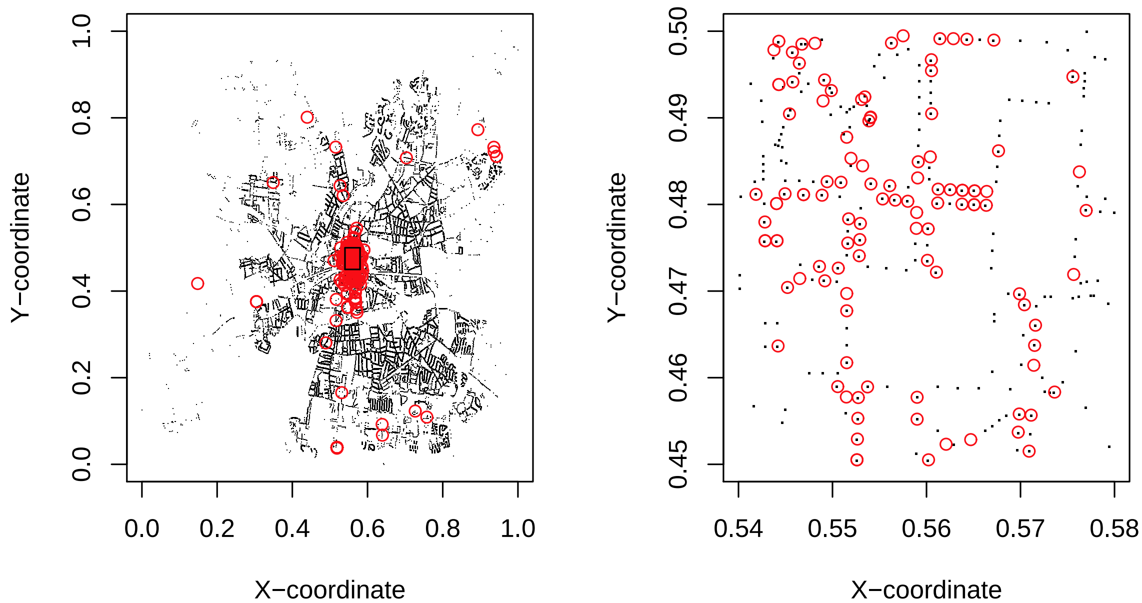

Figure 12: Representation of clothing shops w. r. t. houses in a small center area

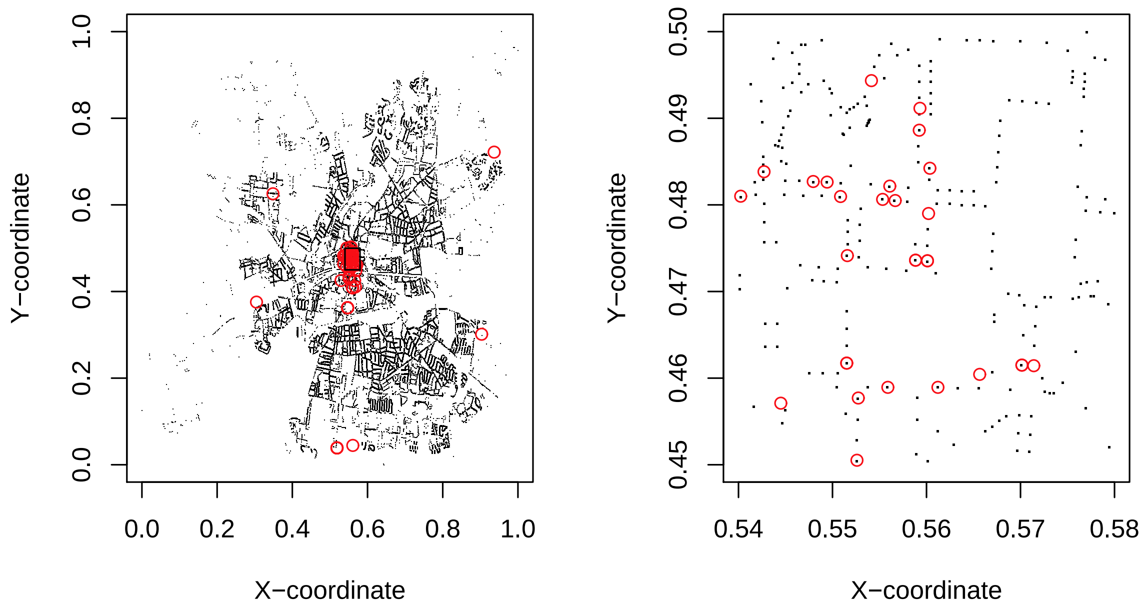

Figure 13: Representation of shoe shops w. r. t. houses in a small center area

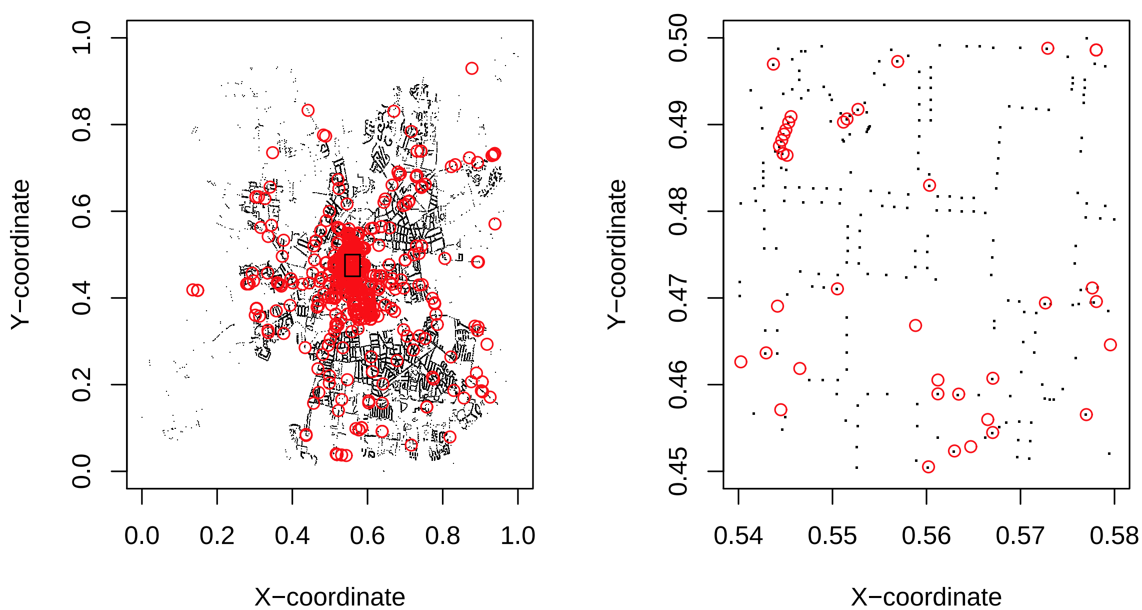

Figure 14: Representation of restaurants w. r. t. houses in a small center area

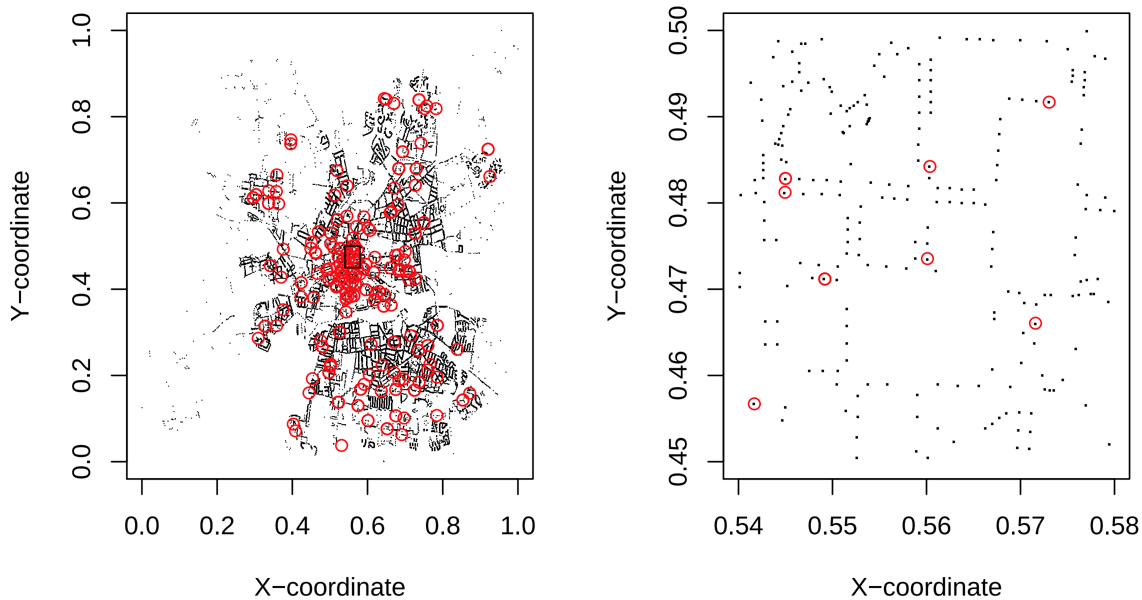

Figure 15: Representation of general medicine doctors w. r. t. houses in a small center area

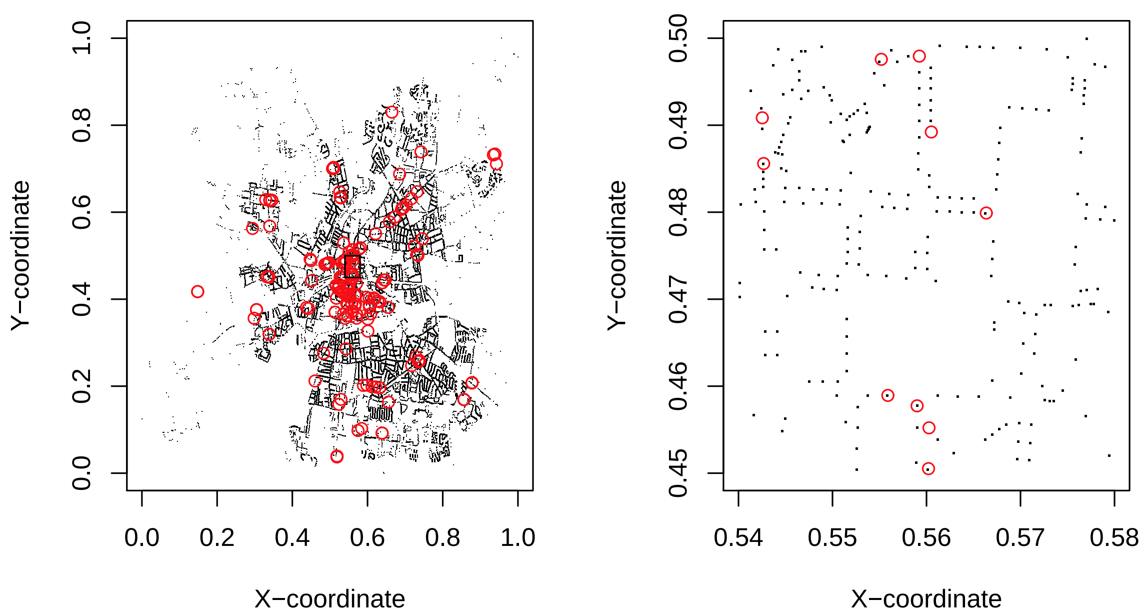

Figure 16: Representation of banks w. r. t. houses in a small center area

The obtained results are given in the following table. Rejection of the null hypothesis for a level of significance $\alpha = 0.05$ is denoted by 1, while acceptance is denoted by 0. Estimations of the p-values of the tests Φ_{Cramer} and Φ_{Bahr} are given between parentheses, and the heuristic choice of σ is given between brackets for the test Φ_{KMMD,n_1,n_2}.

Services	Φ_{Cramer}	Φ_{Bahr}	Φ_{KMMD,n_1,n_2}
Clothing shop (B302)	0(0.16)	0(0.20)	$0[\sigma = 0.01]$
Shoe shop (B304)	1(0.018)	1(0.029)	$1[\sigma = 0.01]$
Restaurant (A504)	0(0.09)	0(0.23)	$1[\sigma = 0.02]$
General medicine (D201)	0(0.18)	0(0.32)	$1\ [\sigma = 0.15]$
Bank (A203)	0(0.51)	0(0.71)	$1[\sigma = 0.14]$

Comments. The three tests give corroborating conclusions for clothing and shoe shops, hence we can trust these conclusions that are: the null hypothesis is accepted for clothing shops while it is rejected for shoe shops for a level of significance $\alpha = 0.05$. These conclusions are in accordance with the intuition we could have just looking at the figures representing the distribution of both shops. As concerns restaurants, the conclusions of the Cramer and Bahr's tests differ from the conclusion of Gretton et al.'s test. This may be explained by the low estimated p-values for the first tests: the test of Gretton et al. rejects the null hypothesis, and with a level $\alpha = 0.1$, the Cramer test would also reject it. As for general medicine and banks, the conclusions of the tests are not corroborating, but one can not here impute this to low p-values for the Cramer and Bahr's test. We conjecture that the tests suffer here from the too small size of the Poisson processes corresponding to these services. Indeed, the whole space $[0, 1]^2$ is considered here, whereas our data are not distributed on this whole space, but on a discrete network of points in this space, or at least on a network of streets. This is again a question to be gone into more deeply, from both practical and theoretical point of views.

4 Conclusions

We have first investigated the performance of the tests proposed by Bahr [1], Baringhaus and Franz [2], Gretton et al. [14] and Fromont, Laurent, Reynaud-Bouret [12] in a simulation study. As expected, when the number of simulated data is nearly the same for the two Poisson processes, but moderately small, the tests are not so powerful. In particular, they all have difficulties to detect the non proportionality or the equality when the intensities of the two Poisson processes only have some very localized differences, even if the test of Fromont, Laurent, Reynaud-Bouret [12] is a bit more powerful in this case. This leads us to think that we surely loose in power by considering spatial Poisson processes in $\mathbb{R}^2$ for our study on representativeness of services. We could for instance develop new tests of comparison for Poisson processes not defined in $\mathbb{R}^2$ but on a space taking the structure of streets or houses into account.

Furthermore, we used these tests to test the proportionality or the equality of the intensities of two Poisson processes representing the rescaled coordinates in $[0, 1]^2$ of two different services of the city of Rennes. We of course sometimes come up against the problem of the sizes of the considered samples again, and the tests seem to be moreover very sensitive to a large difference in sizes for the two processes. This should have to be confirmed by theoretical arguments or by a deeper practical study.

Finally, we used the tests of Bahr [1], Baringhaus and Franz [2], and Gretton et al. [14] to test the proportionality between the intensities of two Poisson processes representing the rescaled coordinates in $[0,1]^2$ of one service on the one side, of houses on the other side. The obtained results are not completely satisfying since we were constrained to consider a very restricted area so that the sizes of the Poisson processes do not differ too much. Using some tests of homogeneity for the Poisson process representing the considered service such as those of Fromont, Laurent, Reynaud-Bouret [11] should be more appropriate. Of course, these tests should also take the structure of the streets or the house to be effective. It is obvious that some tests of homogenity over a convex set of $\mathbb{R}^2$ would not have good performance. Bessac indeed applied such tests of homogeneity derived from a multivariate Kolmogorov-Smirnov tests, and the null hypothesis of homogeneity over convex areas of Rennes was always rejected. Constructing a homogeneity test for a Poisson process defined on a space with a very particular structure will be challenging. The use of kernels as done in learning problems on graphs, phylogenetic trees, images for instance, and as done in [12] should be a pertinent and useful tool.

References

[1] Bahr, R. Ein neuer test fuer das mehrdimensionale zwei-stichproben-problem. *Ph.D. thesis, University of Hanover* (1996).

[2] Baringhaus, L., and Franz, C. On a new multivariate two-sample test. *J. Multivariate Anal. 88*, 1 (2004), 190–206.

[3] Bessac, J. Représentativité spatiale des équipements de la ville de Rennes par rappot aux logements. Technical Report.

[4] Borgwardt, K., Gretton, A., Rasch, M., Kriegel, H.-P., Schölkopf, B., and Smola, A. Integrating structured biological data by kernel maximum mean discrepancy. *Bioinformatics 22 (14)* (2006), e49–e57.

[5] Bovett, J. M., and Saw, J. G. On comparing two Poisson intensity functions. *Comm. Statist. A—Theory Methods 9*, 9 (1980), 943–948.

[6] Chiu, S. N. Parametric bootstrap and approximate tests for two Poisson variates. *J. Stat. Comput. Simul. 80*, 3-4 (2010), 263–271.

[7] Chiu, S. N., and Wang, L. Homogeneity tests for several Poisson populations. *Comput. Statist. Data Anal. 53*, 12 (2009), 4266–4278.

[8] Cox, D. R. Some simple approximate tests for Poisson variates. *Biometrika 40* (1953), 354–360.

[9] Cramer, H. On the composition of elementary errors, 2nd paper. *Skand. Aktuarietidskr. 11* (1928), 141–180.

[10] Deshpande, J. V., Mukhopadhyay, M., and Naik-Nimbalkar, U. V. Testing of two sample proportional intensity assumption for non-homogeneous Poisson processes. *J. Statist. Plann. Inference 81*, 2 (1999), 237–251.

[11] Fromont, M., Laurent, B., and Reynaud-Bouret, P. Adaptive tests of homogeneity for a Poisson process. *Ann. Inst. Henri Poincaré Probab. Stat. 47*, 1 (2011), 176–213.

[12] Fromont, M., Laurent, B., and Reynaud-Bouret, P. The two-sample problem for Poisson processes: adaptive tests with a non asymptotic wild bootstrap approach. *ArXiv* (2012).

[13] Gail, M. Computations for designing comparative poisson trials. *Biometrics 30*, 2 (1974), 231–237.

[14] Gretton, A., Borgwardt, K. M., Rasch, M. J., Schölkopf, B., and Smola, A. A kernel method for the two-sample problem. *J. Mach. Learn. Res. 1* (2008), 1–10.

[15] Himpens, S. Représentativité spatiale du recensement de la population. Technical Report.

[16] Krishnamoorthy, K., and Thomson, J. A more powerful test for comparing two Poisson means. *J. Statist. Plann. Inference 119*, 1 (2004), 23–35.

[17] Møller, J., and Waagepetersen, R. *Statistical Inference and Simulation for Spatial Point Processes*. Monographs in Statistics and Applied Probability. Chapman and Hall/CRC, New York, 2004.

[18] Ng, H. K. T., Gu, K., and Tang, M. L. A comparative study of tests for the difference of two Poisson means. *Comput. Statist. Data Anal. 51*, 6 (2007), 3085–3099.

[19] Praestgaard, J. T. Permutation and bootstrap Kolmogorov-Smirnov tests for the equality of two distributions. *Scand. J. Statist. 22*, 3 (1995), 305–322.

[20] Przyborowski, J., and Wilenski, H. Homogeneity of results in testing samples from Poisson series with an application to testing clover seed for dodder. *Biometrika 31* (1940), 313–323.

[21] Shiue, W.-K., and Bain, L. J. Experiment size and power comparisons for two-sample poisson tests. *J. Roy. Statist. Soc. Ser. C 31*, 2 (1982), 130–134.

[22] Steinwart, I. On the influence of the kernel on the consistency of Support Vector Machines. *J. Mach. Learn. Res. 2*, 1 (2002), 67–93.

[23] van der Vaart, A. W., and Wellner, J. A. *Weak convergence and empirical processes*. Springer Series in Statistics. Springer-Verlag, New York, 1996.